Education

The fundamenal theorem of surfaces theory

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Abstract

The Surface is a branch of differential geometry that plays an important role in practical life. This study aimed to present the fundamental theorem of surface theory. We followed the historical and analytical mathematical, method by providing a general idea of surfaces, the orientation of surfaces, the fundamental theory of surfaces. We found the following some results. There exists an essentially unique surface with specified first and second fundamental forms is a profound result called the fundamental theorem of surface theory.

Keywords: concept of surfaces, orientation of surfaces, first fundamental form, second fundamental form, third fundamental form, the fundamental theorem of surface

1. Introduction

Differential geometry is a branch of geometry that means studying geometric shapes, primarily surfaces the major contribution to the field of differential geometry came in the eighteenth a century by Euler, who identified the basic vectors of surfaces and proved a set of important theorems. It was also the first detailed treatment of the theory of surfaces by Monge. The nineteenth century was the most important milestone in the development of mathematics in general and differential geometry in particular in 1827 Gauss reached a set of properties for surfaces that formed the internal or internal geometry of a surface.Since that time differential geometry has become an indepent branch of mathematics after it was just an application of mathematical analysis. Our object in this paper is to study the Concept of a Surface and We discussed in what sense, and when, it is possible to orient a surface.

2. Concept of a surface

Intuitively we think of a surface as a set of points in space which resembles a portion of a plane in the neighborhood of each of its points. This will be the case if the surface is the image of a sufficiently regular mapping of a set of points in the plane into E^3. Since we want to apply the methods of calculus. We assume that the mapping of is at least of class C^1. Also, in order to insure that the surface has a tangent plane at each point, we assume that the rank of the Jacobian matrix of the mapping is two at each point [7].

Definition (2.1): (Intuitive definition of surface)

A surface is a subset of 1R3 such that each of its points has a neighborhood similar to a piece of a plane that bends smoothly and without self-intersections when bent in three-space [9].

Definition (2.2)

A regular parametric representation of class $C^m (m \geq 1)$ of a set of points S in F^3 is a mapping $X = F(u, v)$ of an open set U in the u, v plane onto S such that

1. F is of class C^m in U.
2. If (e_1, e_2, e_3) is basis in E^3 and $f(u, v) = f_1(u.v)e_1 + f_2(u.v)e_2 + f_3(u, v)e_3$, then for all $(u.v)$ in U.

$$\text{rank} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = 2$$

We recall that F is of class C^m in U if all partial derivative of f of order m or less are continuous in U. [7]

Example (2.3)

The equation

$$x = (u + v)e_1 + (u - v)e_2 + (u^2 + v^2)e_3$$

defines a mapping of the uv plan ont the ellipetic paraboloid $x_3 = \frac{1}{2}(x_1 + x_2)$ shown in Fig No.1 cleanly x has continuous partial derivatives of all orders. Also, for all (u, v).

$$|x_u \times x_v| = \left| \det \begin{pmatrix} e_1 & 1 & 1 \\ e_2 & 1 & -1 \\ e_3 & 2u & 2v \end{pmatrix} \right| = [4 + 8(u^2 + v^2)]^{1/2} \neq 0$$

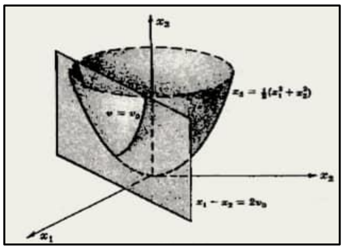

Fig 1: Paraboloid

Thus x is a regular parametric representation of the paraboloid of class C^∞ [7].

3. Orientation of Surfaces

We discussed in what sense, and when, it is possible to orient a surface. Intuitively, since every point p of a regular surface s has a tangent plane $T_p(s)$, the choice of an orientation of $T_p(s)$ induces an orientation in a neighborhood of p, that is, a notion of positive movement along sufficiently small closed carves about each point of the neighborhood. If it is possible to make this choice for each $p \in s$ so that in the intersection of any two neighborhoods the orientations coincide, then s is said to be orientable. If this is not possible, s is called nonorientable [6].

Definition (3.1)

A regular surface S is called orientable if it is possible to cerch it with a family of coordinate neighborhoods in such a way that if a point $p \in s$ belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of such a family is called on orientation of s, and s, in this case is called oriented if such a choice is not possible, the surface is called nonorientable.

Example (3.2)

A surface which is the graph of a differentiable function is an orientable surface. In fact, all surfaces which can be covered by one coordinate neighborhood are trivially orientable [6].

Example (3.3)

The sphere is an orientable surface [6].

4. First Fundamental Form

The first fundamental form, which it based on the metric, encompasses all the intrinsic information about the surface that a 2D inhabitant of the surface can obtain from measurements performed on the surface without appealing to an external dimension. In the old books, the first fundamental form may be la bled as the first fundamental quadratic form.

The first fundamental form is of the length of an element of arc of a curve on a surface is a quadratic expression given by:

$$Is = (ds)^2 = dr.dr = \left(\frac{\partial r}{\partial u^\alpha}\right).\left(\frac{\partial r}{\partial u^\beta}\right) du^\alpha = du^\beta$$

$$= E_\alpha E_\beta du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta$$

$$= E(du')^2 + 2f du' du^2 + G(du^2)^2 \qquad (1)$$

Where E, F, G which in general are continuous variable functions of the surface coordinates u' and u^2, are given by:

$$E = a_{11} = E_1.E_1 = \left(\frac{\partial r}{\partial u^1}\right).\left(\frac{\partial \mathrm{r}}{\partial u^1}\right) = g_{ij}\left(\frac{\partial x^i}{\partial u^1}\right)\left(\frac{\partial x^j}{\partial u^1}\right)$$

$$F = a_{12} = E_1.E_2 = \left(\frac{\partial r}{\partial u^1}\right).\left(\frac{\partial r}{\partial u^2}\right) = g_{ij}\left(\frac{\partial x^i}{\partial u^1}\right)\left(\frac{\partial x^j}{\partial u^2}\right)$$

$$= E_2 E_1 = a_{21}$$

$$G = a_{22} = E_2.E_2 = \left(\frac{\partial r}{\partial u^2}\right).\left(\frac{\partial \mathrm{r}}{\partial u^2}\right) = g_{ij}\left(\frac{\partial x^i}{\partial u^2}\right)\left(\frac{\partial x^j}{\partial u^2}\right)$$

Where the indexed a are the elements of the surface covariant metric tensor, the indexed x are the general coordinates of the enveloping space and g_{ij} is its covariant metric tensor.

For a flat space with a Cartesian coordinate system x^i, the space metric is $g_{ij} = \delta_{ij}$ and hence the above equations become:

$$E = \left(\frac{\partial x^i}{\partial u^1}\right)\left(\frac{\partial x^j}{\partial u^1}\right)$$

$$F = \left(\frac{\partial x^i}{\partial u^1}\right)\left(\frac{\partial x^j}{\partial u^2}\right)$$

$$G = \left(\frac{\partial x^i}{\partial u^2}\right)\left(\frac{\partial x^j}{\partial u^2}\right)$$

The first fundamental form can be cast the following matrix from:

$$I_s = [du^1\ du^2]\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.[E_1\ E_2]\begin{bmatrix} du^1 \\ du^2 \end{bmatrix}$$

$$= [du^1\ du^2]\begin{bmatrix} E_1.E_1 & E_1.E_2 \\ E_2.E_1 & E_2.E_2 \end{bmatrix}\begin{bmatrix} du^1 \\ du^2 \end{bmatrix}$$

$$= [du^1\ du^2]\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} du^1 \\ du^2 \end{bmatrix}$$

$$= [du^1\ du^2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} du^1 \\ du^2 \end{bmatrix} = V\ I_s V^T \qquad (2)$$

Where v is a direction vector, v^T is its transpose, and I_s is the first fundamental form tensor which is equal to the surface covariant metric tensor. Hence, the matrix associated with the first fundamental form is the covariant metric tensor of the surface [11].

Example (4.1)

Consider a surface which is the graph of the function

$$f\,(x.y) = \sqrt{x^2 + y^2}\ (\,(x.y) \neq (0.0)\,).$$

and which is obtained by rotating a half - line $z = x\ (x > 0)$ in the xz - plane about the z - axis. that is. the cone having vertex at the origin.

Using the polar coordinates $(r.\theta)$ given by

$$x = r\cos\theta . y = r\sin\theta.$$

The surface is reparametrized as

$$p\,(r.\theta)\,(r\cos\theta . r\sin\theta . r)\ (r > 0)$$

In which no square root appears. and the map p can be extended smoothly to $r \le 0$ and the image of the singular set $\{r = 0\}$ consists of a point. which is a typical example of a cone - like singularity.

Differentiating this. it holds that

$$dp = (\cos\theta . \sin\theta . 1)\,dr + r\,(-\sin\theta . \cos\theta . 0)\,d\theta.$$

and then the first fundamental form is written as

$$ds^2 = dp.dp = 2dr^2 + r^2 d\theta^2.$$

In particular. the coefficients of the first fundamental form with respect to the polar coordinates $(r.\theta)$ are $E = 2. F = 0$ and $G = r^2$. One can check that the first fundamental form is also expressed using the $(x.y)$ coordinate system as

$$ds^2 = \left(1 + \frac{x^2}{x^2+y^2}\right)dx^2 + \frac{2xy}{x^2+y^2}\,dx\,dy + \left(1 + \frac{y^2}{x^2+y^2}\right)dy^2.$$ [8]

5. The Second Fundamental Form

Let $S \subset R^3$ be an orientable regular surface with smooth unit normal field N Regarded as map between surfaces i.e. $N: s \to s^2$. the map N is also called the Gauss map Let $p \in s$. We consider the differential of N at p

$$\mathrm{d_p N}: \mathrm{T_p s} \rightarrow \mathrm{T_{N(p)} s^2}$$

Now $\mathrm{T_{N(p)} s^2} = N(P)^L = T_p s$. Hence $\mathrm{d_p N}$ is an endomorphism on $\mathrm{T_p s}$.

Definition (5.1)

Let $S \subset R^3$ be a regular surface with orientation given by the unit normal field N. The endomorphism

$$W_p: T_p s \rightarrow T_p s$$
$$W_p(x) = -d_p N(x)$$

is called the weingarten map.

The negative sign appears for historic reasons. If orientation is reversed. i.e. if $-N$ is substituted for N then W also changes is sign [1].

Example (5.2)

Let $s = \{(x.y.0)^T \,|\, x.y \in R\}$ be the $x - y$ plane. $N(x.y.z) = (0.0.1)^T$. Then N is constant and thus $W_p = 0$ for all $p \in s$. [1]

Proposition (5.3)

Let $s \subset R^3$ be an orientable regular surface with weingarten map $W_p: T_p s \rightarrow T_p s. P \in s$. Then W_p is self-adjoint with respect to the first fundamental form

Proof

Let N be the unit normal field of S that induces the Weingarten map. $W_p = -d_p N$. We choose a local parametrisation $(U.F.V)$ at p and set $u = F^{-1}(p)$.

Let

$$x_1 = D_u F(e_1) = \frac{\partial F}{\partial u'}(u)$$

and

$$x_2 = D_u F(e_2) = \frac{\partial F}{\partial u^2}(u)$$

be the corresponding basis vectors of $T_p s$. As N is perpendicular to S everywhere. We have

$$\left\langle \frac{\partial F}{\partial u^2}(u + te_j). N\left(F(u + te_j)\right)\right\rangle \equiv 0$$

Differentiating this equation with respect to t gives

$$0 = \frac{d}{dt}\left\langle \frac{\partial F}{\partial u^i}(u + te_j). N\left(F(u + te_j)\right)\right\rangle\Big|_{t=0}$$
$$= \left\langle \frac{d}{dt}\frac{\partial F}{\partial u^i}(u + te_j)\Big|_{t=0}. N(p)\right\rangle + \left\langle \frac{\partial F}{\partial u^i}\, u. d_p N o D_u F(e_j)\right\rangle$$
$$= \left\langle \frac{\partial^2 F}{\partial u^i} u.. N(p)\right\rangle + \left\langle \frac{\partial^2 F}{\partial u^i \partial u^i}\, u. d_p N o D_u F(e_j)\right\rangle$$

Thus

$$I_p\left(x_i. W_p(x_j)\right) = \langle x_i. W_p(x_j)\rangle = \left\langle \frac{\partial^2 F}{\partial u^j \partial u^i}(u). N(p)\right\rangle \quad (3)$$

By the theorem of Schwarz [10] the two partial derivatives of f can be exchanged and we obtain

$$I_p\left(x_i. W_p(x_j)\right) = \left\langle \frac{\partial^2 F}{\partial u^j \partial u^i}(u). N(p)\right\rangle$$
$$= \left\langle \frac{\partial^2 F}{\partial u^j \partial u^i}(u). N(p)\right\rangle$$
$$= I_p\left(x_j. W_p(x_i)\right)$$

We now know for our basis vectors x_1 and x_2 of $T_p s$ that

$$I_p\left(x_i. W_p(x_j)\right) = I_p\left(x_j. W_p(x_i)\right) = I_p\left(W_p(x_i). x_j\right).$$

Since any two vectors $x.y \in T_p s$ can be written as a linear combination of x_1 and x_2. it immediately follows from the bilinearity of I and the bilinearity of W_p that

$$I_p\left(x. W_p(y)\right) = I_p\left(W_p(x). y\right).$$

i.e. W_p is self-adjoint with respect to I.

Let us recall from linear algebra that if v is a finite - dimensional real vector space with Euclidean scalar product $\langle \ldots \rangle$. then the self-adjoint endomorphisms W on V are uniquely associated with the symmetric bilinear forms B on V. The relation between W and β is $\beta(x.y) = \langle w(x). y\rangle. x.y \in V$.[1]

Definition (5.4)

The second fundamental form in the local coordinates {u. v} (also called the second fundamental form of σ) is the expression

$$F_2 = e du^2 + 2f\, du\, dv + g\, dv^2. \quad (4)$$ [5]

Example (5.5)

Let $S^2 \subset \mathbb{R}^3$. Then $v(p) = p$ and so

$$II(p) = \mathbb{1} - pp^T = \begin{pmatrix} 1 - x^2 & -xy & -xz \\ -yx & 1 - y^2 & -yz \\ -zx & -zy & 1 - z^2 \end{pmatrix}$$

for P = (x. y. z) = S². [3]

6. Third Fundamental Form

If we write

$$C_{\alpha\beta} = g_{ij}\, n_\alpha^i\, n_\beta^i.$$

we see that $C_{\alpha\beta}$ is a symmetric covariant surface tensor of type (0.2) and we call the quadratic form $C \equiv C_{\alpha\beta} du^\alpha\, du^\beta$ the third fundamental form of the surface.

Using Weingarten formula. we get

$$C_{\alpha\beta} = g_{ij}\, n_\alpha^i\, n_\beta^i = g_{ij}\left(-a^{\delta\gamma}\, b_{\delta\alpha}\, x_\gamma^j\right)\left(-a^{\mu\theta}\, b_{\mu\beta}\, x_\theta^j\right)$$
$$= a_{\gamma\theta}\, a^{\delta\gamma}\, b_{\delta\alpha}\, a^{\mu\theta} b_{\mu\beta} = a^{\mu\delta}\, b_{\delta\alpha}\, b_{\mu\beta}$$

This is the relation between three fundamental forms on a surface. but here the third fundamental form is not an actual fundamental form because this can be obtained from first and second fundamental forms [2].

Definition (6.1)

If D is some region on a surface Φ then the real number $w(D) = \iint_D k ds$ is called the integral curvature of D. If D lies entirely in some coordinate neighborhood (u. v) then

$$w(D) = \iint_D k(u.v)\sqrt{E(u.v)G(u.v) - F^2(u.v)}\, dudv$$

Theorem (6.2)

At each point on a regular surface $\emptyset$ of class $c^k (k \geq 3)$ the following curvature.

Proof

Let P be an arbitrary point and $\emptyset$. Introduce coordinates $(u.v)$ in some neighborhood of this point such that the vectors $\vec{r}_u$ and $\vec{r}_v$ at P become parallel to the principal vectors. Then from Rodrigcles's theorem and theorem we obtain at the point P.

$$\langle \vec{r}_u . \vec{r}_v \rangle = 0$$

$$\vec{n}_u = -k_1 \vec{r}_u . \vec{n}_v = -k_2 \vec{r}_v .$$

Map and the Gaussian curvature of $\emptyset$ has the same Singh at each at each point of D [13].

7. The fundamental theorem of surface theory

Suppose we consider a regular oriented surface S and coordinate path v parameterized by $\overrightarrow{x}: U \to R^3$. We have seen that the coefficient (g_{ij}) and (L_{ij}) of the first and second fundamental forms satisfy $\det(g_{ij}) > 0$ and the Gauss – codazzi equations.

That given these conditions. there exists an essentially unique surface with specified first and second fundamental form is a profound result. called the fundamental theorem of surface theory [12].

Theorem (7.1)

If $E.F.G$ and $e.f.g$ are sufficiently differentiable functions of $(u.v)$ that satisfy the Gauss-codazzi equation

$$K = \frac{1}{\det(g_{ij})^2} \left(\begin{vmatrix} -\frac{1}{2} g_{11}, 22 + g_{12}, 12 - \frac{1}{2} g_{11}, 22 & \frac{1}{2} g_{11}, 1 & g_{12}, 1 - \frac{1}{2} g_{11}, 2 \\ g_{12}, 2 - \frac{1}{2} g_{22}, 1 & g_{11} & g_{12} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2} g_{11}, 2 & \frac{1}{2} g_{22}, 1 \\ \frac{1}{2} g_{11}, 2 & g_{11} & g_{12} \\ \frac{1}{2} g_{22}, 1 & g_{21} & g_{22} \end{vmatrix} \right) \quad (4.5)$$

$$\begin{aligned} \frac{\partial L_{11}}{\partial x^2} - \frac{\partial L_{12}}{\partial x^1} &= L_{11}\Gamma_{12}^1 + L_{12}\Gamma_{12}^2 - L_{12}\Gamma_{11}^1 - L_{22}\Gamma_{11}^2 \\ \frac{\partial L_{12}}{\partial x^2} - \frac{\partial L_{22}}{\partial x^1} &= L_{11}\Gamma_{22}^1 + L_{12}\Gamma_{22}^2 - L_{12}\Gamma_{12}^1 - L_{22}\Gamma_{12}^2 \end{aligned} \quad \to (5)$$

and $EG - F^2 > 0$. then there exists a parametrization $\bar{x}$ of a regular orient able surface that admits

$$g_{11} = E . g_{12} = F . g_{22} = G . L_{11} = e . L_{12} = f . L_{22} = g$$

Furthermore. this surface is uniquely determined up to its position in space.

The original proof of this theorem was provided by Bonnet in 1855. In more recent texts. one can find the proof Appendix to Chapter 4 in [7] or in Chapter VI of [4]. We not provide a complete proof of the fundamental theorem of surface theory here since it involves solving a system of partial differential equations. but we sketch the main points behind it. The setup for the proof is to. consider nine functions $\xi_i(u.v)$. $\varphi_i(u.v)$ and $\psi_i(u.v)$ with $1 \leq i \leq 3$. and think of these functions as the components of the vector functions $\vec{x}_u$. and $\vec{N}$ so that $\vec{x}_u = (\xi_1 . \xi_2 . \xi_3)$. $\vec{x}_v = (\varphi_1 . \varphi_2 . \varphi_3)$. $\vec{N} = (\psi_1 . \psi_2 . \psi_3)$. With this setup. the equations that define Gauss's and Weingarten equations. namely. become the following system of 18 partial differential equation for$i = 1.2.3$.

$$\frac{\partial \xi_i}{\partial u} = \Gamma_{11}^1 \xi_i + \Gamma_{11}^2 \varphi_i + L_{11} \psi_i . \frac{\partial \xi_i}{\partial v} = \Gamma_{12}^1 \xi_i + \Gamma_{12}^2 \varphi_i + L_{12} \psi_i$$

$$\frac{\partial \varphi_i}{\partial u} = \Gamma_{21}^1 \xi_i + \Gamma_{21}^2 \varphi_i + L_{21} \psi_i . \frac{\partial \varphi_i}{\partial v} = \Gamma_{22}^1 \xi_i + \Gamma_{22}^2 \varphi_i + L_{22} \psi_i$$

$$\frac{\partial \psi_i}{\partial u} = a_1^1 \xi_i + a_1^2 \varphi_i + L_{21} \psi_i .$$

$$\frac{\partial \psi_i}{\partial v} = a_2^1 \xi_i + a_2^2 \varphi_i \quad (6)$$

In general. when a system of partial differential equations involving n

Functions $u_i(x_1 . \dots . x_m)$ has $n < m$. the solutions may involve not only constants of integration but also unknown functions that can be any continuous function from IR to R (on some appropriate interval). However. when $n > m$. i.e. when there are more functions in the system than there are independent variables. the system may be overdetermined and may either have less freedom in its solution set or have no solutions at all. In fact. one cannot expent the above system to hace solutions if the mixed partial derivatives of $\vec{\xi} = (\xi_1 . \xi_2 . \xi_3)$. $\vec{\varphi} = (\varphi_1 . \varphi_2 . \varphi_3)$. and $\vec{\psi} = (\psi_1 . \psi_2 . \psi_3)$ are not equal. this is usually called the compatibility condition for systems of partial differential equations and as we see in the above systems this condition imposes relations between the functions $\Gamma_{jk}^i(u.v)$ and $L_{jk}(u.v)$.

The key ingredient behind the fundamental theorem of surface theory is theorem V in Appendix B that applied to our context. states that if all second derivatives of the Γ_{jk}^i and L_{jk} functions are continuous and if the compatibility condition holds in equation (4.7). solutions to the system exist and are unique once values for $\vec{\xi}(u_\circ . v_\circ) . \vec{\varphi}(u_\circ . v_\circ)$ and $\vec{\psi}(u_\circ . v_\circ)$ are given. where $(u_\circ . v_\circ)$ is a point in the common domain of $\Gamma_{jk}^i(u.v)$ and $L_{jk}(u.v)$ the compatibility condition required in this theorem is satisfied if and only if the functions $g_{11} = E. g_{12} = F. g_{22} = G. L_{11} = e. L_{12} = f$ and $L_{22} = g$ satisfy the Gauss – codazzi equations. Solution to equation (4.7) can be chosen in such a way that

$\vec{\xi}(u_\circ. v_\circ). \vec{\xi}(u_\circ. v_\circ) = E(u_\circ. v_\circ)\vec{\varphi}(u_\circ. v_\circ). \vec{\varphi}(u_\circ. v_\circ) = G(u_\circ. v_\circ).$
$\vec{\xi}(u_\circ. v_\circ). \vec{\varphi}(u_\circ. v_\circ) = F(u_\circ. v_\circ). \vec{\psi}(u_\circ. v_\circ). \vec{\psi}(u_\circ. v_\circ) = 1$
$\vec{\psi}(u_\circ. v_\circ)\vec{\xi}(u_\circ. v_\circ) = 0. \vec{\psi}(u_\circ. v_\circ). \vec{\varphi}(u_\circ. v_\circ) = 0.$

$$\frac{\vec{\xi}(u_\circ. v_\circ) \times \vec{\varphi}(u_\circ. v_\circ)}{\left\|\vec{\xi}(u_\circ. v_\circ) \times \vec{\varphi}(u_\circ. v_\circ)\right\|} = \vec{\psi}(u_\circ. v_\circ) \quad (7)$$

The next step of the proof is to show that. given the above initial conditions. the following equations hold for all $(u. v)$ where the solution are defined:

$\vec{\xi}(u. v). \vec{\xi}(u. v) = E(u. v)\vec{\varphi}(u. v). \vec{\varphi}(u. v) = G(u. v).$
$\vec{\xi}(u. v). \vec{\varphi}(u. v) = F(u. v). \vec{\psi}(u. v). \vec{\psi}(u. v) = 1$
$\vec{\psi}(u. v)\vec{\xi}(u. v) = 0. \vec{\psi}(u. v). \vec{\varphi}(u. v) = 0.$

$$\frac{\vec{\xi}(u. v) \times \vec{\varphi}(u. v)}{\left\|\vec{\xi}(u. v) \times \vec{\varphi}(u. v)\right\|} = \vec{\psi}(u. v)$$

From the solution for $\vec{\xi}. \vec{\varphi}$ and $\vec{\psi}$. we form the new system of differential equations

$$\begin{cases} \vec{x}_u = \vec{\xi} \\ \vec{x}_v = \vec{\varphi} \end{cases}$$

One easily obtains a solution for the functions $\vec{x}$ over appropriate $(u. v)$ by:

$$\vec{x}(u. v) = \int_{u_\circ}^{u} \vec{\xi}(u. v)\, du + \int_{v_\circ}^{v} \vec{\varphi}(u. v)\, dv \quad (8)$$

The resulting vector function $\vec{x}$ is defined over an open set $U \subset R^2$ containing $(u_\circ. v_\circ)$ and $\vec{x}$ parametrizes a regular surface S. By construction. the coefficients of the first fundamental form this surface are $g_{11} = E. g_{12} = F. g_{22} = G.$

One then proves that it is also true that the coefficients of the second fundamental form satisfy $L_{11} = e. L_{12} = f. L_{22} = g.$

It remains to be shown that this surface is unique up to a rigid motion in IR^3. It is not hard to see that the equalities in equation (4.8) imposed on the initial condition still allow one the freedom to shoose the unit vector $\vec{\xi}(u_\circ. v_\circ) = \frac{\vec{\xi}(u_\circ. v_\circ)}{\left\|\vec{\xi}(u_\circ. v_\circ)\right\|}$ and the vector $\vec{\varphi}(u_\circ. v_\circ)$. which must be perpendicular to $\vec{\xi}(u_\circ. v_\circ)$. The vectors $\vec{\xi}(u_\circ. v_\circ). \vec{\psi}(u_\circ. v_\circ). \vec{\xi}(u_\circ. v_\circ) \times (u_\circ. v_\circ)$.

Form a positive orthonormal frame. so any two choices allowed by equation (4.8) differ from each other by a rotation in IR^3. Finally. the integration in equation (4.9) introduces a constant vector of integration. Thus. two solutions to Gauss's and Weingarten equations differ from each other by a rotation and a ndamental form is not an actual fundamental form because this can be obtained from first and second fundamental forms. Translation namly any rigid motion in IR^3 [12].

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